Write your name here

Surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number	Candidate Number

Further Mathematics

Advanced Subsidiary
Further Mathematics options
22: Further Pure Mathematics 2
(Part of option A only)

Thursday 17 May 2018 – Afternoon

Paper Reference
8FM0/22

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B). Coloured pencils and highlighter pens must not be used.
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- When a calculator is used, the answer should be given to an appropriate degree of accuracy.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- The total mark for this part of the examination is 40. There are 5 questions.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. (i) Using a suitable algorithm and without performing any division, determine whether 23 738 is divisible by 11

(2)

(ii) Use the Euclidean algorithm to find the highest common factor of 2322 and 654

(3)

i) An integer is divisible by 11 iff the sum of its digits with alternating signs is divisible by 11.

$2-3+7-3+8 = 17-6 = 11$ ✓

$\therefore$ 23738 is divisible by 11 //

ii) $2322 = 654(3) + 360$

$654 = 360(1) + 294$

$360 = 294(1) + 66$

$294 = 66(4) + 30$

$66 = 30(2) \boxed{+6}$

$30 = 6(5) + 0$

$\therefore HCF(2322, 654) = \boxed{6}$

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Question 1 continued

(Total for Question 1 is 5 marks)

2.

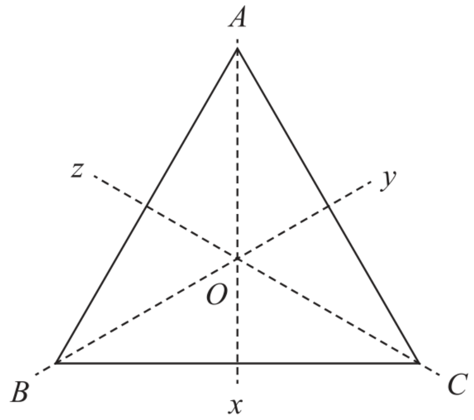

Figure 1

Figure 1 shows an equilateral triangle ABC. The lines x, y and z and their point of intersection, O, are fixed in the plane. The triangle ABC is transformed about these fixed lines and the fixed point O. The lines x, y and z each pass through a vertex of the triangle and the midpoint of the opposite side.

The transformations I, X, Y, Z, R_1 and R_2 of the plane containing triangle ABC are defined as follows:

- I: Do nothing
- X: Reflect in the line x
- Y: Reflect in the line y
- Z: Reflect in the line z
- R_1: Rotate 120° anticlockwise about O
- R_2: Rotate 240° anticlockwise about O

The operation $*$ is defined as 'followed by' on the set $T = \{I, X, Y, Z, R_1, R_2\}$.
For example, $X * Y$ means a reflection in the line x followed by a reflection in the line y.

(a) (i) Complete the Cayley table on page 5

Given that the associative law is satisfied,

(ii) show that T is a group under the operation $*$

(6)

(b) Show that the element R_2 has order 3

(2)

(c) Explain why T is not a cyclic group.

(1)

(d) Write down the elements of a subgroup of T that has order 3

(1)

ii) identity : $I \in T$, so identity is present. ✓

closure : $a * b \in T$ for any choice of $a, b \in T$. (see Cayley Table) ✓

X, Y, Z self-inverse, R_1, R_2 inverses

inverse : all elements have inverses (from Cayley Table) ✓

$\therefore$ T is a group under $*$.

(associativity is assumed)

Question 2 continued

b) $R_2 \neq I$.

$R_2 * R_2 = 480°$ AC rotation $R_2 * R_2 * R_2 = R_1 * R_2$

$R_2 * R_2 * R_2 = 720°$ AC rotation $= 0°$ rotation $= I$

hence R_2 has order 3.

c) There exists no element that generates the group. (No element has order 6)
R_1 & R_2 order 3; X, Y, Z order 2

d) $S = \{I, R_1, R_2\}$

		Second transformation					
	*	I	X	Y	Z	R_1	R_2
First Transformation	I	I	X	Y	Z	R_1	R_2
	X	X	I	R_2	R_1	Z	Y
	Y	Y	R_1	I	R_2	X	Z
	Z	Z	R_2	R_1	I	Y	X
	R_1	R_1	Y	Z	X	R_2	I
	R_2	R_2	Z	X	Y	I	R_1

Turn over for a spare table if you need to re-write your Cayley table

Question 2 continued

Question 2 continued

Only use this grid if you need to re-write your Cayley table

		Second transformation					
	$*$	I	X	Y	Z	R_1	R_2
First Transformation	I						
	X		I			Z	
	Y						
	Z						
	R_1		Y				
	R_2						

(Total for Question 2 is 10 marks)

3 A tree at the bottom of a garden needs to be reduced in height. The tree is known to increase in height by 15 centimetres each year.

On the first day of every year, the height is measured and the tree is immediately trimmed by 3% of this height.

When the tree is measured, before trimming on the first day of year 1, the height is 6 metres.

Let H_n be the height of the tree immediately before trimming on the first day of year n.

(a) Explain, in the context of the problem, why the height of the tree may be modelled by the recurrence relation

$$H_{n+1} = 0.97H_n + 0.15, \quad H_1 = 6, \quad n \in \mathbb{Z}^+$$

(3)

(b) Prove by induction that $H_n = 0.97^{n-1} + 5, \quad n \geqslant 1$

(4)

(c) Explain what will happen to the height of the tree immediately before trimming in the long term.

(1)

(d) By what fixed percentage should the tree be trimmed each year if the height of the tree immediately before trimming is to be 4 metres in the long term?

(2)

a) · The height of the tree at the beginning of a new year (n+1) is 0·97x the height of the previous year, H_n, ∵ it is decreased by 3%.

· The tree then grows 0.15m (15cm) during the course of the year.

· So the height at the end of the year is $0{\cdot}97H_n + 0{\cdot}15$ (which is the same as the height of the tree immediately before trimming on day 1 of the next year.)

∴ $H_1 = 6$ since this is the height of the tree at the start of year 1 before trimming.

b) n=1 : $H_1 = 0{\cdot}97^{1-1} + 5 = 1+5 = 6$
this matches what we are // given.

assume true for n=k : ie $H_{k+1} = 0{\cdot}97^k + 5$

(we want to prove $H_{k+2} = 0.97^{k+1} + 5$)

Question 3 continued

consider $n = k+1$: $H_{k+2} = 0.97H_{k+1} + 0.15$

using the assumption:

$$H_{k+2} = 0.97(0.97^{k} + 5) + 0.15$$
$$= 0.97^{k+1} + 5(0.97) + 0.15$$
$$= 0.97^{k+1} + 5$$

$\therefore$ true for $n = k+1$.

- we have proved the relation true for $n=1$.
- when assumed true for $n=k$ we proved it true for $n=k+1$.
- $\therefore$ By Mathematical Induction the given relation is true for all $n \in \mathbb{Z}^+$

c) $\lim_{n\to\infty}[0.97^{n-1} + 5] = 0 + 5 = 5$

So the height of the tree will approach 5m.

d) $H_{n+1} = kH_n + 0.15$ (like $0.97H_n + 0.15$) k

$\lim_{n\to\infty}[H_n] = 4$.

$$k(4) + 0.15 = 4$$
$$4k = 3.85$$
$$k = \frac{3.85}{4} = 0.9625$$
$$\Rightarrow \boxed{3.75\%}$$

$(1 - 0.9625)$

Question 3 continued

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Question 3 continued

(Total for Question 3 is 10 marks)

4.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$

(7)

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix}$$

find determinant

$$\det(A - \lambda I) = (1-\lambda)(4-\lambda) - (1)(-2) = 0$$

$$\Rightarrow 4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0$$

$\therefore \lambda = 3, \lambda = 2$ are eigenvalues of A

$$\therefore \boxed{D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}$$

To find eigenvectors, consider $Ax = \lambda x$:

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \quad (\lambda = 2)$$

$\Rightarrow x + y = 2x \quad \therefore x = y \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.

Question 4 continued

$$\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \qquad (\lambda = 3)$$

$\Rightarrow x + y = 3x \quad \therefore \quad 2x = y \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 2$.

$$\therefore \boxed{P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}} //$$

($D = \text{diag}(\lambda_1, \lambda_2)$ & matrix that diagonalises A, P, is eigenvectors of A)

(Total for Question 4 is 7 marks)

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5. A complex number z is represented by the point P on an Argand diagram.

Given that $\arg\left(\dfrac{z-6i}{z-3i}\right)=\dfrac{\pi}{3}$

$\arg(z-6i)-\arg(z-3i)$

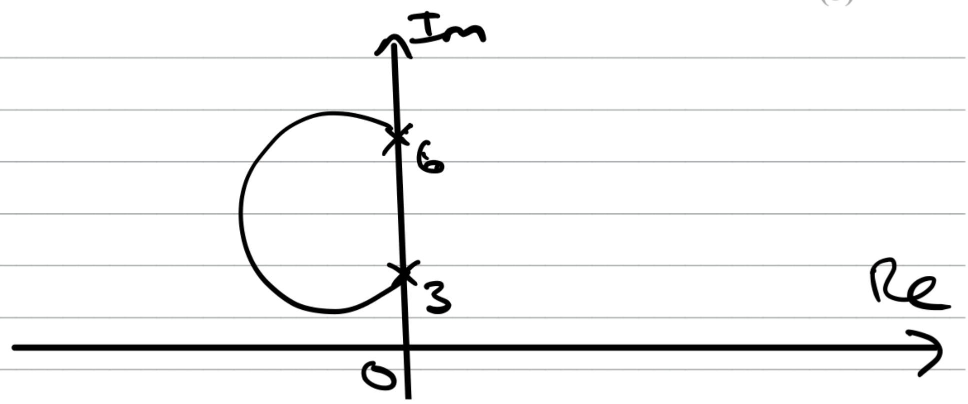
(a) sketch the locus of P as z varies,

(3)

(b) find the exact maximum possible value of $|z|$

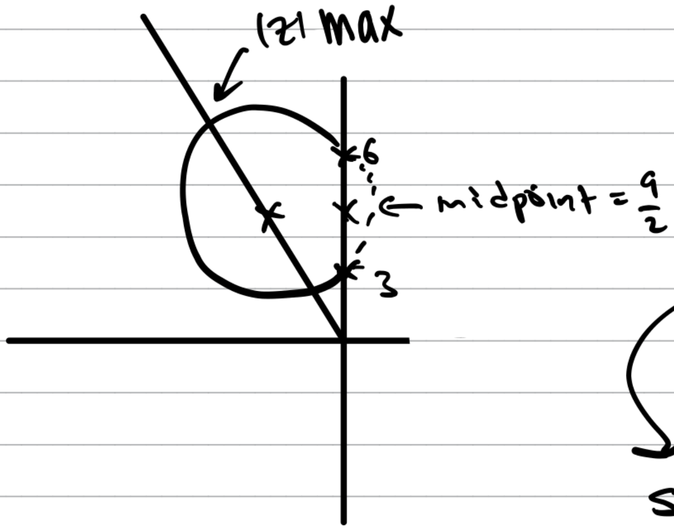
(5)

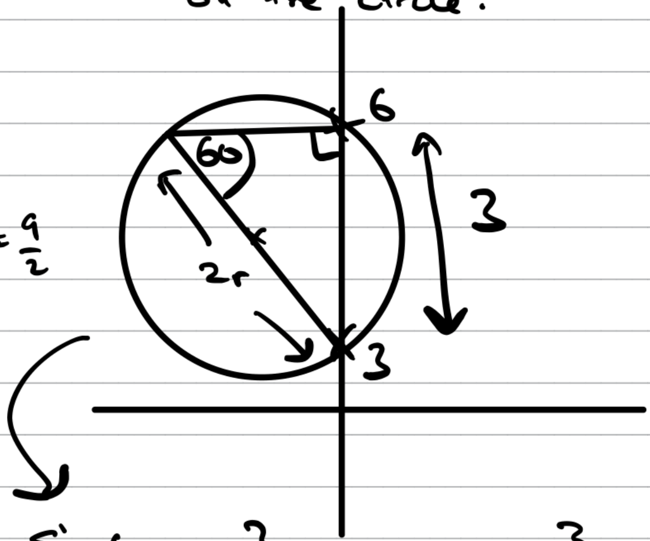
a)



b)

consider when $\arg(z-3i)$ passes through the centre of the circle:





$\sin 60 = \dfrac{3}{2r}$ $\therefore r = \dfrac{3}{2\sin 60}$

$= \sqrt{3}$

$\Rightarrow$ Centre of circle has coordinates $(-\sqrt{3}\cos 60, \frac{9}{2})$

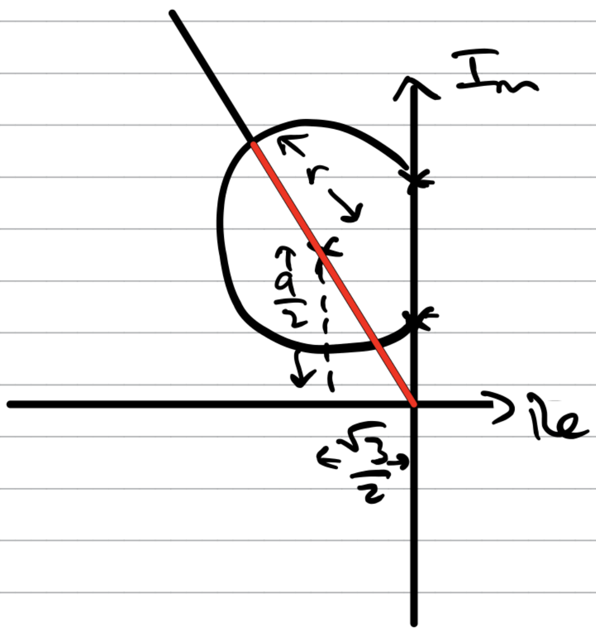
$= \left(-\frac{\sqrt{3}}{2}, \frac{9}{2}\right)$

Question 5 continued

$\therefore$ distance from O to centre of circle is

$$\sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{9}{2}\right)^2} = \sqrt{21}$$

$$\therefore |z|_{max} = \sqrt{21} + r = \boxed{\sqrt{21}+\sqrt{3}}$$



alternatively: $\dfrac{-1.5}{\tan\left(\frac{\pi}{3}\right)} = -\dfrac{\sqrt{3}}{2} =$ x-coord of centre

$$r = \frac{1.5}{\sin\left(\frac{\pi}{3}\right)}$$

$$d = \sqrt{4.5^2+0.75}+\sqrt{3}$$

$$= \sqrt{21}+\sqrt{3}$$

Question 5 continued

(Total for Question 5 is 8 marks)

TOTAL FOR FURTHER PURE MATHEMATICS 2 IS 40 MARKS